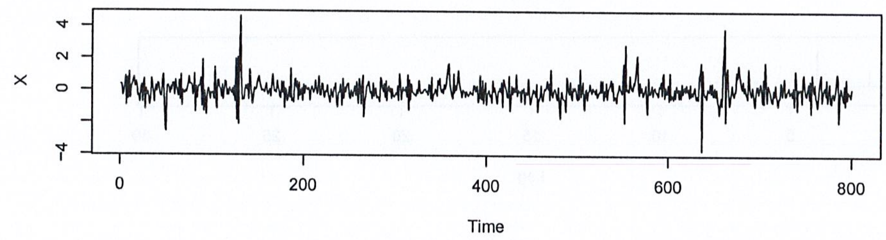
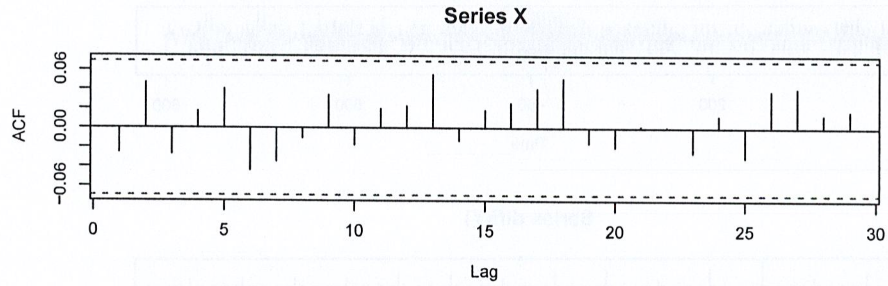
Final exam

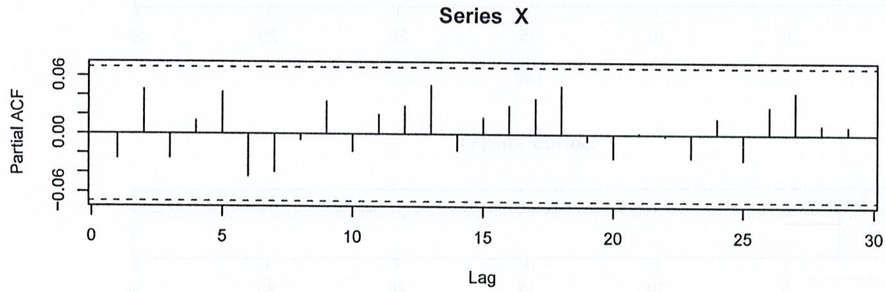
Duration 2h. Documents allowed. Computers and calculators are NOT allowed. All answers should be justified.

Exercise 1. Below are the plots of three times series X, Y and Z, together with related diagrams. Propose a model for each of these series. Tell when the diagrams allow to estimate some of the parameters in the model (and give estimated values).

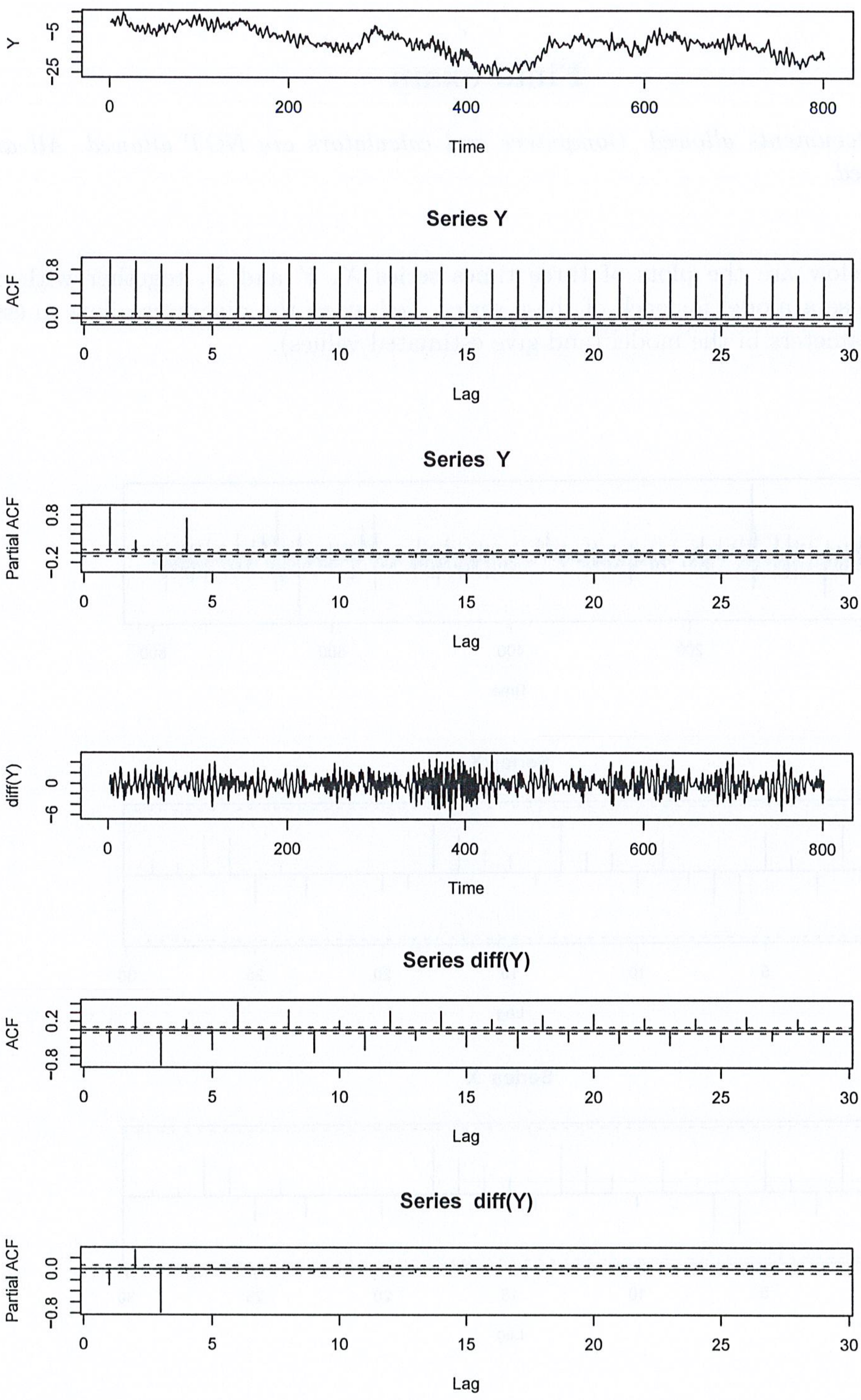
1. Series X :



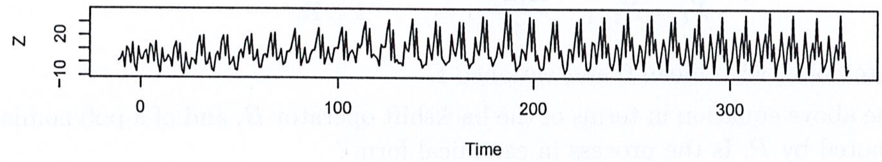


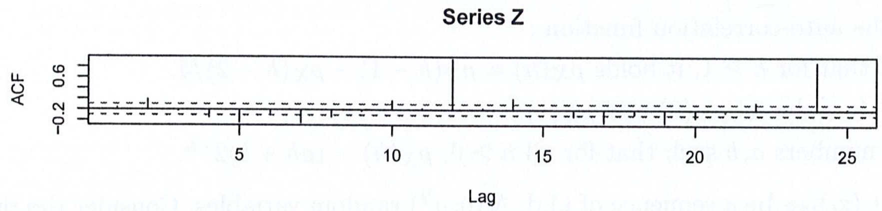


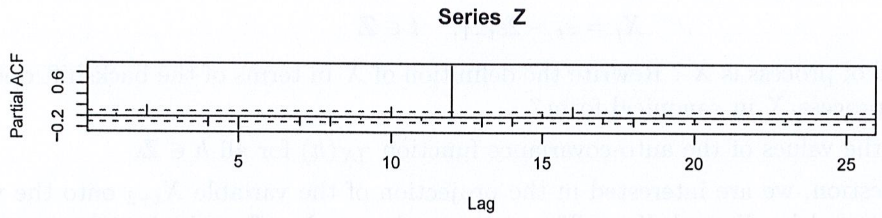
2. Series Y :


Y
Time
Series Y
ACF
Lag
Series Y
Partial ACF
Lag
diff(Y)
Time
Series diff(Y)
ACF
Lag
Series diff(Y)
Partial ACF
Lag

3. Series Z :





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> DZ=diff(Z,lag=12)
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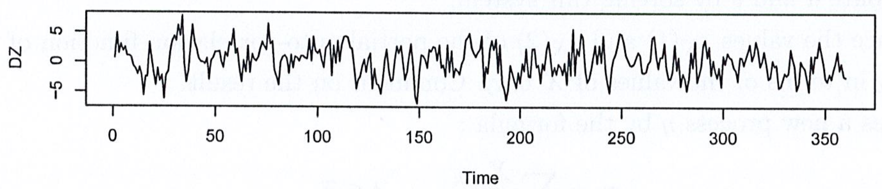

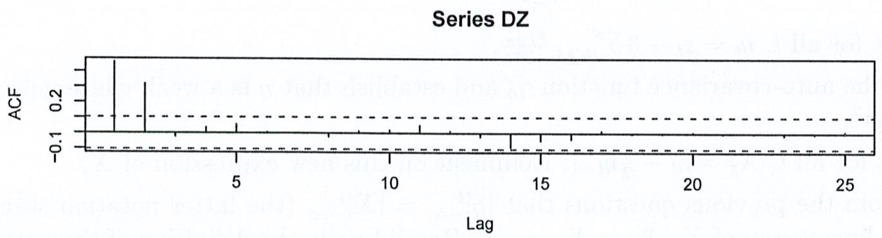

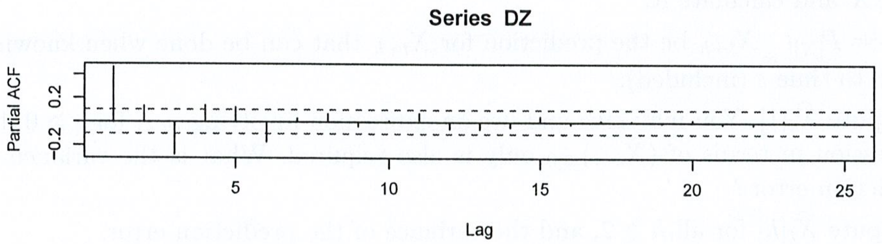

Exercise 2. We study an auto-regressive process X defined by the following recursion

$$X_t = X_{t-1} - \frac{X_{t-2}}{4} + \varepsilon_t, \qquad t \in \mathbb{Z},$$

where ε is a white noise, with mean 0 and variance 1.

1. Express the above equation in terms of the backshift operator B, and of a polynomial that will be denoted by P. Is the process in canonical form ?
2. According to the theory, what is the value of $\mathbb{E}(X_{t-h}\varepsilon_t)$ for $h > 0$?
3. Study of the auto-correlation function :
   (a) Show that for $h \geq 1$, it holds $\rho_X(h) = \rho_X(h-1) - \rho_X(h-2)/4$.
   (b) Give the value of $\rho_X(0)$ and $\rho_X(1)$.
   (c) Find numbers a, b such that for all $h \geq 0$, $\rho_X(h) = (ah + b)2^{-h}$.

Exercise 3. Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a sequence of i.i.d. $\mathcal{N}(0, \sigma^2)$ random variables. Consider the process X defined by

$$X_t = \varepsilon_t - 2\varepsilon_{t-1}, \quad t \in \mathbb{Z}$$

1. What kind of process is X ? Rewrite the definition of X in terms of the backshift operator B. Is the process X in canonical form ?
2. Compute the values of the auto-covariance function $\gamma_X(h)$ for all $h \in \mathbb{Z}$.
3. In this question, we are interested in the projection of the variable X_{t+2} onto the vector space generated by X_t and X_{t+1}. There are numbers $a, b \in \mathbb{R}$ such that this projection can be written as

$$P_{[X]_t^{t+1}} X_{t+2} = aX_{t+1} + bX_t.$$

   (a) Give a system of equations that a and b should verify.
   (b) Compute a and b by solving this system.
   (c) Deduce the values $\tau_X(1)$ and $\tau_X(2)$ of the partial auto-correlation function of X.
4. Express ε_t in terms of the values of X only. Comment on the result.
5. One defines a new process η by the formula :

$$\eta_t = \sum_{k\geq 0} \frac{X_{t-k}}{2^k}, \qquad t \in \mathbb{Z}.$$

   Prove that for all t, $\eta_t = \varepsilon_t - 3\sum_{k\geq 1} \frac{\varepsilon_{t-k}}{2^k}$.
6. Compute the auto-covariance function γ_η and establish that η is a weak white noise, with variance $4\sigma^2$.
7. Show that for all t, $X_t = \eta_t - \frac{1}{2}\eta_{t-1}$. Comment on this new expression of X.
8. Deduce from the previous questions that $[\eta]_{-\infty}^t = [X]_{-\infty}^t$ (the latter notation stands for the closed linear span of $X_t, X_{t-1}, X_{t-2}, \ldots$). Recall briefly the definition of the innovation process of X and calculate it.
9. Let $\widehat{X}_t[h] := P_{[X]_{-\infty}^t} X_{t+h}$ be the prediction for X_{t+h} that can be done when knowing the process up to time t (included).
   (a) Compute $\widehat{X}_t[1]$. You may give and use an expression involving η_{t-i} for $i \geq 0$, but an expression in terms of $(X_{t-i})_{i\geq 0}$ only is also required. What is the variance of the prediction error ?
   (b) Compute $\widehat{X}_t[h]$ for all $h \geq 2$, and the variance of the prediction error.