- *The two exercises are independent.*
- *Number of points for each question is given into brackets.*
- *Total number of points = 20.*

EXERCISE 1. We consider a sample $(X1, \ldots, X_n)$ of i.i.d. random variables with density

$$f_\theta(x) = \frac{1}{6\theta^4} x^3 \exp(-\frac{x}{\theta}) \boldsymbol{I}_{\mathbb{R}_+}(x), \quad \theta > 0.$$

1. [2 points] Determine a sufficient and complete statistic for θ.
2. [2 points] Given that $\mathbb{E}(X) = 4\theta$ and $\mathbb{V}ar(X) = 4\theta^2$, find the unique optimal unbiased estimator $\widehat{\theta}$ of θ.
3. [2 points] Show that $\widehat{\theta}$ is asymptotically Gaussian.
4. [3 points] Calculate the Fisher information $I_{X_1}(\theta)$, and conclude whether $\widehat{\theta}$ is asymptotically efficient.

EXERCISE 2. Let X be a *single* observation from the density parameterized by $\theta \in [-1, 1]$

$$f_\theta(x) = 2\theta x + 1 - \theta,$$

for $x \in [0, 1]$, and 0 otherwise. We consider the family of tests with critical region $C_K = \{X \geq K\}$. In questions 1 to 3 below, we want to test $H_0 : \theta = 0$ against $H_1 : \theta = 1$.

1. [2 points] What is the p-value of the observation $x = 2/3$?
2. [2 points] Prove that the family of tests with critical region C_K defined above is the family of most powerful tests for testing H_0 against H_1.
3. [2 points] What is the most powerful test of size $\alpha = 0.05$?

   We now consider testing $H_0 : \theta \leq 0$ against $H_1 : \theta > 0$.
4. [2 points] Compute the power and the size of the test with critical region C_K for $K = 1/2$.
5. [3 points] What is the family of most powerful tests in this situation ?