MARTINGALES AND APPLICATIONS
EXAMINATION- SECOND SESSION
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YEAR : 2017 - M1 TSE

COURSE : O. FAUGERAS

Closed book. Exercises are independent.
It is advised to provide careful reasoning and justifications in your answers. It will be taken a great care of them in the notation.

Exercise 1 Let X_n be a $(\mathcal{F}_n)_{n\geq 0}$ martingale. Set

$$Y_n = \begin{cases} X_n - 1 & \text{if } X_n - 1 \geq 2 \\ 2 & \text{if } X_n - 1 < 2 \end{cases}$$

Show that (Y_n) is a sub-martingale.

Exercise 2 Let $0 < \theta < 1$ a parameter. Let $(X_n)_{n\in\mathbb{N}}$ a stochastic process defined as $X_0 = x$ a.s. with $0 < x < 1$, and for $n > 0$,

$$X_{n+1} = \theta X_n + (1-\theta)\epsilon_{n+1},$$

where $(\epsilon_n)_{n\geq 1}$ is a sequence of $0, 1$-valued r.v. s.t. conditionally on $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, ϵ_{n+1} is Bernouilli distributed with parameter X_n (i.e. $P(\epsilon_{n+1} = 1|\mathcal{F}_n) = X_n = 1 - P(\epsilon_{n+1} = 0|\mathcal{F}_n)$).

1. Show that for every $n \geq 0$, $0 < X_n < 1$ a.s.
2. Show that $(X_n)_{n\in\mathbb{N}}$ is a $\mathcal{F}_n$-martingale.
3. Show that (X_n) converges a.s. and in L_2 to some r.v. X_∞
4. Show that for every $n \geq 0$,

$$E(X_{n+1} - X_n)^2 = (1-\theta)^2 E[X_n(1-X_n)]$$

5. Deduce that $E[X_\infty(1-X_\infty)] = 0$
6. Determine the law of X_∞.