Toulouse School of Economics
École d'économie de Toulouse

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Master 1 Economics, Econometrics Statistics

Epreuve : Martingales Theory & Applications

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MARTINGALES AND APPLICATIONS
EXAMINATION- FIRST SESSION
2 PAGES
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Closed book. Exercises are independent.
It is advised to provide careful reasoning and justifications in your answers. It will be taken a great care of them in the notation.

Exercise 1 Conditional expectation
Let X, Y be two $\mathcal{N}(0,1)$ independent random variables. Set $Z = X + Y$.

1. Compute $E(Z|X)$
2. Compute $E(X|Z)$.
3. Give an alternative demonstration of the previous result.

Exercise 2 Sequential Games and Ruin Probability
Players A and B play repeatedly in a fair game of Heads and Tails : A's initial fortune is $a > 0$, $a \in \mathbb{N}$, B's initial fortune is $b > 0$, $b \in \mathbb{N}$. At each turn, if A wins, his fortune increases by 1, and B's fortune decreases by 1. The game ends whenever one of the player is ruined. Let $(\epsilon_i)_{i=1,2,\ldots}$ an i.i.d. sequence of r.v. defined on some probability space $(\Omega, \mathcal{A}, P)$ with

$$P(\epsilon_1 = 1) = P(\epsilon_1 = -1) = \frac{1}{2},$$

standing for the result of the successive tosses, so that A's fortune at time $n \in \mathbb{N}$ writes

$$X_n = a + \sum_{i=1}^{n} \epsilon_i.$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$.

1. Show that $(X_n, n \in \mathbb{N})$ is a $(\mathcal{F}_n)$-martingale
2. For $c > 0$, let $T_c := \inf\{n \in \mathbb{N}, X_n = c\}$. Let $u(a)$ the probability that A is ruined before B and T the duration of the game.
   (a) Express $u(a)$ and T as functions of T_0, T_{a+b}.
   (b) Show that T is a $(\mathcal{F}_n)-$stopping time.
   (c) Assume $ET < \infty$. Compute $u(a)$ using optional stopping.
3. Let $M_n = X_n - nE\epsilon_1^2$.
   (a) Show that $(M_n, n \in \mathbb{N})$ is a $(\mathcal{F}_n)$-martingale.
   (b) Use martingales techniques to compute ET.

Exercise 3 A fundamental economic process : Ponzi schemes
A Ponzi scheme is a fraudulent investment operation where the operator, an individual or organization, pays returns to its investors from new capital paid to the operators by new investors, rather than from profit earned through legitimate sources. Operators of Ponzi schemes usually entice new investors by offering higher returns than other investments, in the form of short-term returns that are either abnormally high or unusually consistent.

One modelizes the evolution of a population of customers in a Ponzi/pyramid scheme as follows : One starts with one gullible customer. Such initial customer recruits(lures) $X \in \mathbb{N}$ new customers in the fraudulent operation. In turn, these X new customers recruits each a random number of customers, and so on.. Let $(X_k^{(n)}), n, k \in \mathbb{N}$ a doubly infinite array of i.i.d. r.v. with discrete distribution X. Assume $P(X = 0) > 0$. (i.e. there is positive probability of exiting the Ponzi chain). The population Z_n at time n is given by

$$\begin{aligned} Z_0 &= 1 \\ Z_{n+1} &= \sum_{k=1}^{Z_n} X_k^{(n+1)} \end{aligned}$$

At time n, $X_k^{(n+1)}$ is the number of new customers which will be recruited in the $n+1$th generation by the k-th investor (if there is one) in the nth generation. We will assume X is distributed according to a Geometric$(0 < p < 1)$ distribution :

$$P(X = k) = pq^k, k \in \mathbb{N}, \quad q := 1 - p$$

Set also $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(X_k^{(m)}, k \in \mathbb{N}^*, m \leq n)$.

1. Set $L_X(\theta) := E[\theta^X]$, $L_X : \theta \in [0,1] \to [0,1]$ the Moment generating function of X. Compute L_X, EX.
2. Compute $E[Z_{n+1}|\mathcal{F}_n]$.
3. Let $M_n := \frac{Z_n}{(EX)^n}$. Deduce that$(M_n, n \in \mathbb{N})$ is a $\mathcal{F}_n$-martingale.
4. Deduce the asymptotic behavior of M_n. In which case, $p > 1/2$ or $p < 1/2$, do you have extinction with probability one ?
5. Let $L_n(\theta) := E[\theta^{Z_n}]$ be the Moment generating function of Z_n. Find a recursion between L_{n+1} and L_n. Deduce the expression of L_n in terms of L_X.
6. Let $B_n = \{Z_n = 0\}$ the event "The Ponzi-scheme collapses at time n", and π_n the corresponding probability, $\pi_n := P(B_n)$. The extinction probability is

$$\pi := P(\exists n \in \mathbb{N}, Z_n = 0) = \lim_{n \uparrow \infty} \uparrow \pi_n$$

   Show that π satisfy the equation

$$\pi = L_X(\pi)$$

7. Compute the possible values of π for the Geometric distribution. Use results of question 4) to deduce the unique value of π if $p < 1/2$ or if $p > 1/2$.