Toulouse School of Economics

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ECOLE D'ÉCONOMIE DE TOULOUSE

Année universitaire 2016-2017

Session 1

Semestre 1

M1 Economics
M1 Economics & Statistics

Epreuve : Dynamic Optimization

Date de l'épreuve : 27 Mars 2017

Durée de l'épreuve : 1h30

liste des documents autorisés : résumé de cours

liste des matériels autorisés :

nombre de pages : 4

Proof

Do the proof of the following lemma and theorem :

Lemma.
Let X, Γ, F *and* β *satisfy* **H 2**. *Then, for any* $x_0 \in X$ *and any* $\underline{x_0} = (x_0, x_1, \cdots, x_t, \cdots) \in \Pi(x_0)$

$$u(\underline{x_0}) = F(x_0, x_1) + \beta u(\underline{x_1})$$

where $\underline{x_1} = (x_1, \cdots, x_t, \cdots)$.

Theorem. *Let* X, Γ, F *and* β *satisfy* **H 1** *and* **H 2**, *then the function* v^* *satisfies the functional equation*

$$v^*(x) = \sup_{y \in \Gamma(x)} \{\, F(x,y) \, + \, \beta v^*(y) \,\}, \qquad \forall x \in X$$

Problem

Take $0 < \beta \leq 1$ and consider the following sequence problem :

$$(SP) \quad \left\| \begin{array}{l} \displaystyle\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sqrt{x_t - x_{t+1}}, \\ \\ \text{such that } 0 \leq x_{t+1} \leq x_t, \\ \text{given } x_0 \geq 0. \end{array} \right.$$

1. **Connection between the sequence problem and the functional equation.**

   For any feasible plan $\underline{x_0} = (x_0, x_1, ., x_t, .) \in \Pi(x_0)$, the cost along the plan is

$$u(\underline{x_0}) = \sum_{t=0}^{\infty} \beta^t \sqrt{x_t - x_{t+1}}.$$

   The value function of the problem (SP) is

$$v^*(x_0) = \max_{\underline{x_0} \in \Pi(x_0)} u(\underline{x_0}).$$

   (a) Show that the value function v^* of the problem (SP) satisfies the functional equation :

$$(FE) \qquad v(x) = \sup_{0 \leq y \leq x} \left\{ \sqrt{x-y} + \beta v(y) \right\} \qquad \forall x \in \mathbb{R}_+$$

(b) Show that an optimal plan $\underline{x_0}^* = (x_0^*, x_1^*, ., x_t^*, .) \in \Pi(x_0^*)$ satisfies

$$v^*(x_t^*) = \sqrt{x_t - x_{t+1}} + \beta v^*(x_{t+1}^*).$$

(c) Show that the value function v^* satisfies :

$$\forall x_0 \geq 0 \qquad \sqrt{x_0} \leq v^*(x_0).$$

(Hint : built a plan $\underline{x_0} \in \Pi(x_0)$ such that $u(\underline{x_0}) = \sqrt{x_0}$)

(d) Show that, for $0 < \beta < 1$, the value function v^* satisfies :

$$\forall x_0 \geq 0 \qquad \sqrt{x_0} \leq v^*(x_0) \leq \frac{\sqrt{x_0}}{1-\beta}.$$

2. **Solution of the functional equation**

We look for a solution of the functional equation in the following Banach space :

$$\mathcal{H}(\mathbb{R}^+) = \Big\{ f : \mathbb{R}^+ \to \mathbb{R}, \text{ continuous, homogeneous of degree } \tfrac{1}{2} \text{ and s. t.,} \\ \exists M \, \forall x \in \mathbb{R}^+ \, |f(x)| \leq M(\sqrt{x}) \Big\},$$

endowed with the norme $\|f\| = \sup_{\mathbb{R}^{*+}} \dfrac{|f(x)|}{\sqrt{x}} = \max_{\|X\|=1} |f(x)|$.

Let us define the operator T on $\mathcal{H}(\mathbb{R}^+)$ by

$$Tf(x) = \sup_{0 \leq y \leq x} \left\{ \sqrt{x-y} + \beta f(y) \right\} \qquad \forall x \in \mathbb{R}^+$$

(a) Show that if $f \in \mathcal{H}(\mathbb{R}^+)$ then $Tf \in \mathcal{H}(\mathbb{R}^+)$.

(b) Show that T satisfies

a. (monotonicity) f, $g \in \mathcal{H}(\mathbb{R}^+)$ and $f \leq g$ implies $T(f) \leq T(g)$.

b. (discounting) for all $f \in \mathcal{H}(\mathbb{R}^+)$, $a \geq 0$.

$$T(f + a(\sqrt{.})) \leq T(f) + \beta a \sqrt{.}$$

Then deduce that, for $0 < \beta < 1$, T is a contraction with modulus β.

3. **Application**

We suppose here that $0 < \beta < 1$.

(a) Show that the functional equation (FE) has a unique solution.

(b) Show that this solution is concave.

(c) Show that the optimal policy is a function.

(d) Show that the functional equation has a solution of the type for all $x \geq 0$, $f(x) = k\sqrt{x}$ with the constant $k \in \mathbb{R}^+$.

(e) Explain why this solution of the functional equation is the value function and find the optimal policy function.

4. **Case** $\beta = 1$

   (a) Lets take for all $x \in \mathbb{R}^+$, $v_0(x) = \sqrt{x}$. Define the sequence $v_{n+1} = Tv_n$. Show that for all $x \in \mathbb{R}^+$, $v_n(x) = \sqrt{nx}$. Deduce from 1.(c) the value function.