

Université Toulouse 1 Capitole
École d'économie de Toulouse

Année universitaire 2016-2017

Session 1

Semestre 1

Master 1 Econométries, Statistics – Decision Mathematics

Epreuve : Optimization

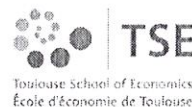
Date de l'épreuve : 12 décembre 2016

Durée de l'épreuve : 1h30

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ANNÉE UNIVERSITAIRE 2015-2016
SEMESTRE 2 DU MASTER 1 "ECONOMICS" & "ECO-STAT"
ADVANCED CALCULUS

Lundi 12 décembre 2016 – 14h-15h – 2 pages

Exercise 1

Consider

$$\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$
$$b : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto 2x_1y_1 + x_2y_2 + 2x_3y_3 - x_1y_2 - x_2y_1 + x_1y_3 + x_3y_1$$

1. Prove that b is scalar product in $\mathbb{R}^3$.
2. Determine the matricial representation of b in the canonical basis.
3. Give an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^3$ for the scalar product b .
4. Determine the matricial representation of b in the basis $\mathcal{B}$.
5. What is the associated duality operator?
6. Determine a basis of the orthogonal to $F := \{(x, y, z) \in \mathbb{R}^3 : y = x\}$.
7. Determine the orthogonal projection on F .
8. What is the distance of $(1, 1, 1254)$ to F .
9. What is the distance of $(1, 0, 1)$ to F .
10. What is the distance of $(1, 0, 1)$ to the orthogonal of F .
11. Consider

$$(\mathbb{R}^3)^2 \times (\mathbb{R}^3)^2 \rightarrow \mathbb{R}$$
$$m : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto 2b(x_1, y_1) + b(x_2, y_2) + 2b(x_3, y_3) - b(x_1, y_2) - b(x_2, y_1) + b(x_1, y_3) + b(x_3, y_1)$$

- (a) Prove that m defines a scalar product in $\mathbb{R}^3 \times \mathbb{R}^3$.
- (b) What is the associated duality operator.

Exercise 2

Let K be a closed subset of X and f a lower semi-continuous function defined on the subspace K .
Consider

$$f_K : \begin{array}{ll} X & \rightarrow (0, \infty) \\ x & \mapsto f(x) + \chi_K(x) \end{array}$$

where

$$\chi_K : \begin{array}{ll} X & \rightarrow (0, \infty) \\ x & \mapsto \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{otherwise} \end{cases} \end{array}$$

Prove that f_K is lower semi-continuous.

Exercise 3

Let $(f_i)_{i \in I}$ be a family of convex function and $(\alpha_i)_{i \in I}$ be reals numbers. Prove that

$$\{x \in X : f_i(x) \leq \alpha_i, i \in I\}$$

is convex.

Exercise 4

Let l be a scalar product in $\mathbb{R}^n$ and N the associated norm. Let $p \in X^*$. Consider

$$\forall x \in X, \quad f(x) = \frac{1}{2}N(x)^2 + \langle p, x \rangle$$

and the minimisation problem

$$\inf_{x \in X} f(x) . \tag{1}$$

1. Prove that if $\bar{x} \in X$ is a solution to (1) then

$$\forall y \in X, \quad \langle p + L\bar{x}, y \rangle \geq 0 .$$

Hint: consider elements of the form $x = \bar{x} + \theta y$.

2. Prove that a necessary and sufficient condition for $\bar{x} \in X$ to be a solution to (1) is that

$$L\bar{x} + p = 0 .$$

3. Prove that if $p \notin \text{Im}L$ then

$$\inf_{x \in X} f(x) = -\infty .$$