TSE

Toulouse School of Economics
École d'économie de Toulouse

UNIVERSITÉ TOULOUSE 1 CAPITOLE

Université Toulouse 1 Capitole
Ecole d'économie de Toulouse

Année universitaire 2016-2017

Session 1

Semestre 1

Master 1 Econométrics, Statistics – Decision Mathematics
Epreuve : Optimization
Date de l'épreuve : 12 décembre 2016
Durée de l'épreuve : 1h30

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ANNÉE UNIVERSITAIRE 2015-2016
SEMESTRE 2 DU MASTER 1 "ECONOMICS" & "ECO-STAT"
ADVANCED CALCULUS

Lundi 12 décembre 2016 – 14h-15h – 2 pages

Exercise 1

Consider

$$b : \begin{array}{ccc} \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} & \mapsto & 2x_1y_1 + x_2y_2 + 2x_3y_3 - x_1y_2 - x_2y_1 + x_1y_3 + x_3y_1 \end{array}$$

1. Prove that b is scalar product in $\mathbb{R}^3$.
2. Determine the matricial representation of b in the canonical basis.
3. Give an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^3$ for the scalar product b.
4. Determine the matricial representation of b in the basis $\mathcal{B}$.
5. What is the associated duality operator?
6. Detrmine a basis of the orthogonal to $F := \{(x, y, z) \in \mathbb{R}^3 \, : \, y = x\}$.
7. Determine the orthogonal projection on F.
8. What is the distance of $(1, 1, 1254)$ to F.
9. What is the distance of $(1, 0, 1)$ to F.
10. What is the distance of $(1, 0, 1)$ to the orthogonal of F.
11. Consider

$$m : \begin{array}{ccc} (\mathbb{R}^3)^2 \times (\mathbb{R}^3)^2 & \to & \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} & \mapsto & 2b(x_1, y_1) + b(x_2, y_2) + 2b(x_3, y_3) - b(x_1, y_2) - b(x_2, y_1) + b(x_1, y_3) + b(x_3, y_1) \end{array}$$

   (a) Prove that m defines a scalar product in $\mathbb{R}^3 \times \mathbb{R}^3$.

   (b) What is the associated duality operator.

Exercise 2

Let K be a closed subset of X and f a lower semi-continuous function defined a the subspace K. Consider

$$f_k : \begin{array}{ccc} X & \to & (0, \infty) \\ x & \mapsto & f(x) + \chi_K(x) \end{array}$$

where

$$\chi_K : \begin{array}{ccc} X & \to & (0, \infty) \\ x & \mapsto & \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{otherwise} \end{cases} \end{array}$$

Prove that f_K is lower semi-continuous.

Exercise 3

Let $(f_i)_{i\in I}$ be a family of convex function and $(\alpha_i)_{i\in I}$ be reals numbers.
Prove that

$$\{x \in X \,:\, f_i(x) \leq \alpha_i, i \in I\}$$

is convex.

Exercise 4

Let l be a scalar product in $\mathbb{R}^n$ and N the associated norm. Let $p \in X^*$. Consider

$$\forall x \in X\,, \quad f(x) = \frac{1}{2}N(x)^2 + \langle p, x\rangle$$

and the minimisation problem

$$\inf_{x\in X} f(x)\,. \tag{1}$$

1. Prove that if $\bar{x} \in X$ is a solution to (1) then

$$\forall y \in X\,, \quad \langle p + L\bar{x}, y\rangle \geq 0\,.$$

   Hint: consider elements of the form $x = \bar{x} + \theta y$.

2. Prove that a necessary and sufficient condition for $\bar{x} \in X$ to be a solution to (1) is that

$$L\bar{x} + p = 0\,.$$

3. Prove that if $p \notin \mathrm{Im}L$ then

$$\inf_{x\in X} f(x) = -\infty\,.$$