Toulouse School of Economics
École d'économie de Toulouse

UNIVERSITÉ TOULOUSE 1 CAPITOLE

Université Toulouse 1 Capitole
Ecole d'économie de Toulouse

Année universitaire 2016-2017

Session 1

Semestre 1

Master 1 Econometrics, Statistics

Epreuve : Mathematical Statistics 1

Date de l'épreuve : 13 décembre 2016

Durée de l'épreuve : 2h

Liste des documents autorisés : english dictionary

Liste des matériels autorisés : calculator

Nombre de pages (y compris page de garde) : 3

MATHEMATICAL STATISTICS
2016-2017
Final exam
WITHOUT document

Justifications of the results are at least as important as the results themselves in the grading.

Question 1.
Define the empirical influence function and give the shape of its graph for the case of the sample mean and sample median

Question 2.
Write the optimization problems defining the theoretical and the empirical quartile regression (order 1/4)

Question 3.
Explain how parametric bootstrap allows to estimate the bias and the variance of an estimator when closed formulas are not available

Exercise 1

Let X be a real random variable with the following *p.d.f.* :

$$f_X(x) = \frac{1}{x^2}\mathbb{1}_{[1,+\infty)}.$$

1) Find the cumulative distribution fonction, F_X, of X.
2) Does the expectation of X exist ?
Let $(X_n)_{n\geqslant 1}$ be a sequence of *i.i.d.* real random variables distributed like X. We set

$$Y_n = \frac{1}{n}\max(X_1, \dots, X_n)$$

3) Give the support of Y_n.
4) Compute the cumulative distribution function, F_n, of Y_n.
5) Show that $(Y_n)_n$ converges in distribution to Y, where Y has the following *c.d.f.* :

$$F_Y(t) = \begin{cases} \exp(-\frac{1}{t}) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

6) Let $Z = \frac{1}{Y}$. Give a *p.d.f.* of Z. Bonus question : Do you recognise any common distribution ?

Exercise 2

Let $X_1, \cdots, X_n$ be a sample of size n of the continuous distribution with density parameterized by $\theta > 0$:

$$f_\theta(x) = \theta \frac{\exp(-\theta x)}{(1 + \exp(-\theta x))^2}.$$

1) compute the quantile of order α of this distribution.

2) let $R_n = \hat{q}_{9/10} - \hat{q}_{1/10}$ be the difference between the sample deciles of order 0.9 and 0.1 respectively. Derive the asymptotic distribution of R_n

3) Bonus question : Given that the non centered moments of order 2 and 4 (same as centered since the mean is zero) of this distribution are $\mu_2 = \frac{\pi^2}{3\theta^2}$ and $\mu_4 = \frac{7\pi^4}{15\theta^4}$, what is the asymptotic distribution of $S_n = \frac{1}{n}\sum_{i=1}^n X_i^2$?

Exercise 3

The gamma distribution with parameters $b > 0$ and $c > 0$ $G(b, c)$ is given by its p.d.f.

$$f(x) = (\frac{x}{b})^{c-1} \frac{1}{b\Gamma(c)} \exp(-x/b),$$

for $x > 0$ and where Γ is the Euler Gamma function. Its expected value is equal to bc and its variance to b^2c.

Let $X_1, \cdots, X_n$ be a sample of size n from the Poisson distribution with mean $\theta > 0$. The loss function is quadratic loss.

1) Compute the Bayesian estimator $\hat{\theta}_B$ of θ for the prior distribution given by the gamma distribution $G(1, 2)$

Consider the following family of estimators $\hat{\theta}_\alpha = \alpha\bar{X} + 2(1 - \alpha)$.

2) compute the risk of $\hat{\theta}_\alpha$

3) compute the Bayesian risk of $\hat{\theta}_\alpha$ for the prior of question 1

4) find the parameter α that minimizes the Bayesian risk of $\hat{\theta}_\alpha$. Comment your result.