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Ecole d'économie de Toulouse

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Session 1

Semestre 1

Master 1 Economics & Statistics

Epreuve : Decision Mathematics 1

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Exercise 1. Consider the zero-sum game $G = (X, Y, g)$, where $X = Y = [0, 1]$, and

$$\forall x \in X, \forall y \in Y,\ g(x, y) = (x - y)^2.$$

An interpretation is that both players choose a location in $[0, 1]$, player 1 wants to be far from player 2, and player 2 wants to be close to player 1.

1.a) Does G have a value in pure strategies ?

1.b) Show that G has a value v in mixed strategies.

1.c) Suppose that Player 1 plays the mixed strategy $\sigma = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ (i.e., play $x = 0$ with probability $1/2$ and play and $x = 1$ with probability $1/2$). Compute the best replies of player 2.

1.d) Compute v and give an optimal mixed strategy for each player.

Exercise 2. Given x and y in $[0, 1]$, we write $I(x, y) = \{t \in [0, 1], |t - x| < |t - y|\}$, and we denote by $\lambda(I(x, y))$ the length of the interval $I(x, y)$.

Consider the zero-sum game $G = (X, Y, g)$, where $X = Y = [0, 1]$, and

$$\forall x \in X, \forall y \in Y,\ g(x, y) = \begin{cases} 1 & \text{if } \lambda(I(x, y)) > \lambda(I(y, x)) \\ -1 & \text{if } \lambda(I(x, y)) < \lambda(I(y, x)) \\ 0 & \text{if } \lambda(I(x, y)) = \lambda(I(y, x)) \end{cases}.$$

An interpretation is that both players are politicians choosing their political program in $[0, 1]$. The population is represented by a continuum of voters uniformly distributed on $[0, 1]$, and each voter t will vote for the candidate whose program is closer to t. The politician with the most votes will win the election and have a payoff of 1.

Show that G has a value in pure strategies and give an optimal pure strategy for each player.

Exercise 3. Consider the following dynamic game with vector payoff, where $I = \{T, B\}$, $J = \{L, R\}$, and the vector payoff is given by:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (0,1) & (-1,1) \\ (1,-1) & (0,1) \end{pmatrix} \end{array}$$

For each of the following sets, is it approachable by player 1 ?

$$C_1 = \mathbb{R} \times \{1\},\ C_2 = \{(t, -t), t \in \mathbb{R}\},\ C_3 = \{0\} \times \mathbb{R}.$$

Exercise 4. In this exercise we fix $I = J = \{1, 2\}$.

A) Let z be in $\mathbb{R}^4$. Compute the orthogonal projection $\pi_C(z)$ of z onto $C = \mathbb{R}^4_-$, and show that $< z - \pi_C(z), \pi_C(z) >= 0$.

B) We view $\mathbb{R}^4$ as the set $\mathbb{R}^{2\times 2}$ of matrices $r = (r_{i'i''})_{(i',i'')\in I\times I} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$, and define the non positive orthant $C = \mathbb{R}^{2\times 2}_-$. We consider the game with vector payoff with actions sets $I = J$, and the payoff in $\mathbb{R}^{2\times 2}$ is given by: $\forall i \in I, \forall j \in J$, $r(i,j) = (r(i,j)_{i'i''})_{(i',i'')\in I\times I}$, with

$$r(i,j)_{i',i''} = \begin{cases} \mathbb{1}_{i''=j} - \mathbb{1}_{i'=j} & \text{if } \ i = i' \\ 0 & \text{if } \ i \neq i' \end{cases}$$

So for instance $r(1,1) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, and $r(2,1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

B.1) Compute $r(1,2)$ and $r(2,2)$.

B.2) Show that for any z in $\mathbb{R}^{2\times 2}$, there exists $x \in [0,1]$ such that for all j in J :

$$< x\, r(1,j) + (1-x)\, r(2,j) - \pi_C(z), z - \pi_C(z) >= 0.$$

C) Consider a decision-maker, who has to select at each stage n some action i_n in I. The environment (nature, adversary, other agents following their own goals) will select a sequence $(j_n)_{n\geq 1}$ with values in J. At each stage n the choices of the decision-maker and of the environment are supposed simultaneous, and at the end of each stage n the decision-maker observes j_n and receives the payoff 1 if $i_n = j_n$, and 0 if $i_n \neq j_n$.

Give an explicit description of a strategy of the decision-maker with no internal regret.