Université Toulouse 1 Capitole
Ecole d'économie de Toulouse

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Session 1

Semestre 1

Master 1 Economics & Statistics

Epreuve : Advanced Analysis

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Année 2016/2017 UNIVERSITÉ TOULOUSE I CAPITOLE

Advanced Analysis

Terminal Exam - December 2016 **Advanced Analysis**

Exercice 1. [10 Points]

We consider

$$\ell^1 = \left\{ (u_n)_{n\geq 1} \mid \sum_{n\geq 1} |u_n| < +\infty \right\}$$

and define the ℓ^1 norm

$$\|u\|_1 := \sum_{n\geq 1} |u_n|.$$

1. Show briefly that ℓ^1 is a normed vector space.
2. Show that $(\ell^1, \|.\|_1)$ is a Banach space.
3. We consider $f : \ell^1 \longrightarrow \ell^1$ the function defined by

$$\forall u \in \ell^1 \qquad f(u) = (1 - \sum_{n\geq 1} |u_n|, u_0, \ldots, u_n, \ldots).$$

   Prove that f is a continuous function.
4. Compute $\|f(u)\|_1$ when $u \in B_{\ell^1}(0,1) = \left\{ v : \sum_{n\geq 1} |v_n| \leq 1 \right\}$. Deduce that $f(B_{\ell^1}(0,1)) \subset B_{\ell^1}(0,1)$.
5. Is it true that $f(B_{\ell^1}(0,1)) = B_{\ell^1}(0,1)$.
6. Show that $B_{\ell^1}(0,1)$ is closed and convex.
7. Can we apply the Brouwer theorem? Why?
8. Show that f does not have any fixed point.

Exercice 2. [8 Points]

We consider h and v two continuous applications from $[-1,1]$ into the domain

$$\mathcal{R} = \{(x,y) \mid a \leq x \leq b,\, c \leq y \leq d\}.$$

We assume that:

- the x-coordinate of $h(-1)$ is equal to a while the x-coordinate of $h(1)$ is equal to b.
- the y-coordinate of $v(-1)$ is equal to c while the y-coordinate of $v(1)$ is equal to d.

In what follows, we will denote $h = (h_1, h_2)$ and $v = (v_1, v_2)$.

1. Represent the situation with a nice picture.
2. We aim to show that the two curves defined by h and v have an intersection. Translate this assumption into an analytical criterion.
3. We consider the application $F : [a,b] \times [c,d] \longrightarrow \mathbb{R}^2$ defined by

$$\forall (t,s) \in [a,b] \times [c,d] \qquad F(t,s) = \left(\frac{v_1(s) - h_1(t)}{N(t,s)}, \frac{h_2(t) - v_2(s)}{N(t,s)} \right)$$

   with $N(t,s) = |v_1(s) - h_1(t)| \vee |h_2(t) - v_2(s)|$. We assume that the two curves have no intersection. Show that in that case, F is a continuous function over $[-1,1]^2$.

4. Show that F has a fixed point (t_0, s_0).
5. Show that the fixed point is located on the boundary of $\mathcal{R}$.
6. Obtain a contradiction and conclude that the two curves have an intersection.

Exercice 3. [6 Points]

We consider E an euclidean space such that $dim(E) < +\infty$ and an application $P : E \longrightarrow E$ such that a radius $\rho > 0$ exists such that

$$\forall x \in \partial\mathcal{B}(0,\rho) \qquad \langle P(x), x\rangle \geq 0.$$

Above, $\partial\mathcal{B}(0,\rho)$ refers to the sphere of radius ρ.

1. Draw an illustration in 2-D.
2. We assume that P does not vanish on $\overline{B(0,\rho)}$ and define g as

$$\forall x \in \overline{B(0,\rho)} \qquad g(x) = -\frac{\rho}{\|P(x)\|}P(x).$$

   Prove that g is continuous.
3. Prove that g has a fixed point.
4. Obtain a contradiction and conclude that P vanishes. Explain on your picture.