UNIVERSITÉ TOULOUSE 1 CAPITOLE

TSE
Toulouse School of Economics
École d'économie de Toulouse

Année Universitaire 2012-2013
SESSION 1

MASTER 1

PROBABILITY THEORY
(durée 1h30)

Mardi 15 janvier 2013 ~ 10h00 -11h30

J. P. DECAMPS

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Documents not allowed

1. Let $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ be a probability space where $\Omega = \{1, 2, 3, 4, 5, 6\}$, and where $\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \frac{1}{16}$, $\mathbb{P}(\{3\}) = \mathbb{P}(\{4\}) = \frac{1}{4}$ and $\mathbb{P}(\{5\}) = \mathbb{P}(\{6\}) = \frac{3}{16}$. Let X and Y two random variables defined on Ω by the relations $X = 2\mathbb{1}_{\{1,2\}} + 8\mathbb{1}_{\{3,4,5,6\}}$ and $Y = 4\mathbb{1}_{\{1,2,3\}} + 6\mathbb{1}_{\{4,5,6\}}$. Compute $\mathbb{E}[X|Y = 4]$, $\mathbb{E}[X|Y]$.

2. Let us consider a financial model with two dates 0 and 1. At each date an entrepreneur decides whether to take his private firm public (this decision is called an IPO decision). The IPO decision is made based on a cutoff rule: an IPO takes place if the firm's expected profitability exceeds a given cutoff. Let $c \in R$ denotes the cutoff, which is known, and $\overline{\Theta}$ the firm's average profitability, which is unknown.

   At time 0, the entrepreneur's beliefs about the firm's average profitability is that $\overline{\Theta}$ has a normal distribution $\mathcal{N}(\hat{\theta}_0, \hat{\sigma}_0)$.

   At time 1, the entrepreneur observes a signal about average profitability $\overline{\Theta}$, namely the realized profitability θ. The realized profitability θ corresponds to the value taken by the random variable $\Theta = \overline{\Theta} + U$ where U is independent from Θ and has a Normal distribution $\mathcal{N}(0, \sigma_U)$.

   (a) Write as a function of $\hat{\theta}_0$, $\hat{\sigma}_0$, σ_U and θ the expected profitability at time 1 $\hat{\theta} = E[\overline{\Theta} \mid \overline{\Theta} + U = \theta]$ (you can use directly a result proven in the course).

   (b) Show that, for an IPO to take place at time 1, realized profitability θ must exceed expected profitability $\hat{\theta}$ (Note that, if an IPO takes place at date 1, it did not take place at time 0).

3. (a) Let X be a non-negative random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that, for any $\alpha > 0$,

   $$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}.$$

   (b) Let X be a non-negative random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{E}[X] < \infty$. Show that $\mathbb{P}(X = \infty) = 0$
   (Hint: write $(X = \infty) = \cap_{n\geq 1}(X \geq n)$)

   (c) Let X be a random variable defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with finite mean μ. Show that

   $$\mathbb{P}(|X - \mu| \geq \alpha) \leq \frac{\text{Var}(X)}{\alpha^2}.$$

   (d) Let X and Y two random variables in $L^1(\Omega, \mathcal{A}, \mathbb{P})$

   i. Give a characterization of $\mathbb{E}[X|Y]$.

   ii. Let us consider two σ-fields $\mathcal{F}$ and $\mathcal{G}$ with $\mathcal{F} \subset \mathcal{G} \subset \mathcal{A}$. Show that $\mathbb{E}[\mathbb{E}[Y|\mathcal{G}]|\mathcal{F}] = \mathbb{E}[Y|\mathcal{F}]$.